LEFT-ORDERABILITY AND EXCEPTIONAL DEHN SURGERY ON TWO-BRIDGE KNOTS

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Abstract. We show that any exceptional non-trivial Dehn surgery on a hyperbolic two-bridge knot, yields a 3-manifold whose fundamental group is left-orderable. This gives a new supporting evidence for a conjecture of Boyer, Gordon and Watson.

1. Introduction

A group G is *left-orderable* if it admits a strict total ordering $<$, which is invariant under left-multiplication. The fundamental groups of many 3-manifolds, for example, all knot and link groups, are known to be left-orderable. On the other hand, there are many 3-manifolds whose fundamental groups are not left-orderable. Since a left-orderable group is torsion-free, lens spaces provide such typical examples. There is a more general notion, called an L-space, introduced by Ozsváth and Szabó [15] in terms of Heegaard Floer homology. These include lens spaces, elliptic manifolds, etc. Recently, Boyer, Gordon and Watson [2] conjectured that a prime, rational homology 3-sphere is an L-space if and only if its fundamental group is not left-orderable. This conjecture is verified for a few classes of 3-manifolds [2, 9, 11].

In [16], the second author proved that any exceptional non-trivial Dehn surgery on a hyperbolic twist knot yields a 3-manifold whose fundamental group is left-orderable. Since such a twist knot does not admit Dehn surgery yielding an L-space, it gives a supporting evidence for the conjecture of Boyer, Gordon and Watson.

In the present paper, we examine the other hyperbolic two-bridge knots. According to the classification of exceptional Dehn surgery on hyperbolic two-bridge knots [4], it is sufficient to consider the following three cases; twist knots, $K[c_1, c_2]$ (c_1 and c_2 are even, and $|c_1|, |c_2| > 2$), and $K[c_1, c_2]$ (c_1 is odd, c_2 is even, and $|c_1|, |c_2| > 2$). Here, a two-bridge knot $K[c_1, c_2]$ corresponds to a (subtractive) continued fraction

$$[c_1, c_2]^- = \cfrac{1}{c_1 - \cfrac{1}{c_2}}$$

in the usual way ([10]). See also Section 2. In particular, the double branched cover of the 3-sphere S^3 branched over $K[c_1, c_2]$ is a lens space $L(c_1c_2 - 1, c_2)$. The first case was settled in [16]. For the second case, the only exceptional non-trivial surgery is 0-surgery. The resulting manifold is prime ([7]) and has positive Betti

2010 *Mathematics Subject Classification.* Primary 57M25; Secondary 06F15.

Key words and phrases. left-ordering, two-bridge knot, Dehn surgery.

The first author is partially supported by an NSERC postdoctoral fellowship. The second author is partially supported by Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research (C), 22540088.

number, so its fundamental group is left-orderable [3]. For the last case, the only exceptional non-trivial surgery has slope $2c_2$, which yields a toroidal manifold. We settle this remaining case.

Theorem 1.1. *Let K be the two-bridge knot corresponding to a (subtractive) continued fraction $[c_1, c_2]$, where c_1 is odd and c_2 is even, and $|c_1|, |c_2| > 2$. Then $2c_2$-surgery on K yields a graph manifold whose fundamental group is left-orderable.*

Hence this immediately implies the following.

Corollary 1.2. *Let K be a hyperbolic two-bridge knot. Then any exceptional non-trivial Dehn surgery on K yields a 3-manifold whose fundamental group is left-orderable.*

We would expect that any non-trivial Dehn surgery on a hyperbolic two-bridge knot yields a 3-manifold whose fundamental group is left-orderable, but this is still a challenging problem.

2. L-space surgery

Let K be the two-bridge knot corresponding to $[c_1, c_2]$, satisfying the assumption of Theorem 1.1. Set $c_1 = 2b_1 + 1$ and $c_2 = 2b_2$. We can assume that $c_1 > 0$, so $b_1 \geq 1$, and $|b_2| \geq 2$. In Figure 1, a rectangular box means half-twists with indicated numbers. They are right-handed if the number is positive, left-handed, otherwise.

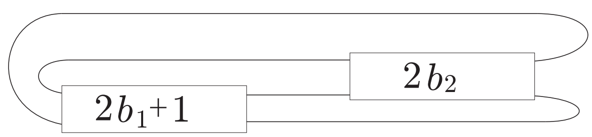

Figure 1. The two-bridge knot $K[2b_1 + 1, 2b_2]$

Since 2-bridge knots are alternating ([8]), we can invoke Theorem 1.5 of [15] to conclude that $2c_2$-surgery on K does not yield and L-space. However, we can argue this fact directly as follows.

Lemma 2.1. *The knot $K = K[2b_1 + 1, 2b_2]$ is fibered if and only if $b_1 = 1$ and $b_2 > 0$.*

Proof. We have

$$[2b_1+1, 2b_2]^- = \begin{cases} [2b_1, \underbrace{-2, -2, \dots, -2}_{2b_2-1}]^- & \text{if } b_2 > 0, \\ [2b_1+2, \underbrace{2, 2, \dots, 2}_{-2b_2-1}]^- & \text{if } b_2 < 0. \end{cases}$$

This implies that a minimal genus Seifert surface of K is obtained by plumbing a single $2b_1$-twisted, or $(2b_1 + 2)$-twisted, annulus with Hopf bands. Then the conclusion immediately follows from [6]. □

Recall that a rational homology 3-sphere Y is an *L-space* if its Heegaard Floer homology $\widehat{HF}(Y)$ has rank equal to $|H_1(Y; \mathbb{Z})|$.

Proposition 2.2. *The knot K does not admit an L-space surgery.*

Proof. By [14], if K is not fibered, then K does not admit an L-space surgery. Hence it is sufficient to examine the case where $b_1 = 1$ and $b_2 > 0$ by Lemma 2.1. Then, as seen in the proof of Lemma 2.1, K has genus b_2.

On the other hand, the double branched cover of S^3 branched over K is a lens space $L(6b_2 - 1, 2b_2)$. Hence the determinant $|\Delta_K(-1)|$ of K equals to $6b_2 - 1$, where $\Delta_K(t)$ is the Alexander polynomial of K.

Suppose that K admits an L-space surgery. Then $\Delta_K(t)$ has a form of

$$\Delta_K(t) = (-1)^k + \sum_{j=1}^{k} (-1)^{k-j}(t^{n_j} + t^{-n_j})$$

for some sequence of positive integers $0 < n_1 < n_2 < \cdots < n_k$ by [15]. Since K is fibered, its genus equals to n_k. Thus $|\Delta_K(-1)| \leq 2k + 1 \leq 2n_k + 1$. Hence, $6b_2 - 1 \leq 2b_2 + 1$, a contradiction. $\square$

3. Fundamental group

By using the Montesinos trick ([12]), we will examine the structure of the resulting manifold by $4b_2$-surgery on $K = K[2b_1 + 1, 2b_2]$ to obtain a presentation of its fundamental group.

First, put the knot K in a symmetric position as illustrated in Figure 2. By

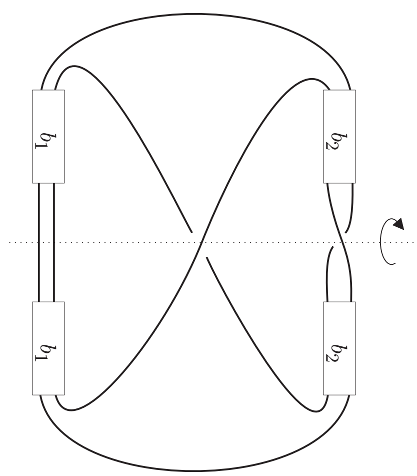

Figure 2. K in a symmetric position

taking a quotient under the involution, whose axis is indicated by a dotted line there, we obtain a 2-string tangle $\mathcal{T}$, which is drawn as the outside of a small circle, in Figure 3.

If the ∞-tangle, which is indicated there, is filled into the small circle, then we obtain a trivial knot. This means that the double branched cover of the tangle $\mathcal{T}$ recovers the exterior of K. We chose the framing so that the 0-tangle filling corresponds to $4b_2$-surgery on K upstairs. Figure 4 shows the resulting link by filling the 0-tangle. The link admits an essential Conway sphere S depicted there.

Let $\mathcal{T}_1 = (B_1, t_1)$ and $\mathcal{T}_2 = (B_2, t_2)$ be the tangles defined by S, that are located outside and inside of S, respectively. Here, t_1 consists of two arcs, but t_2 consists of

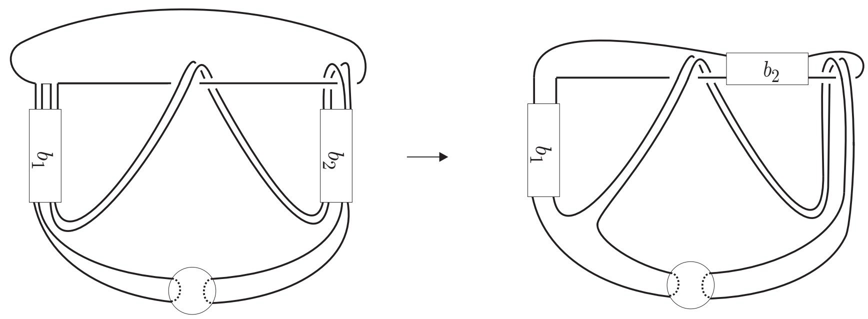

FIGURE 3. Montesinos trick

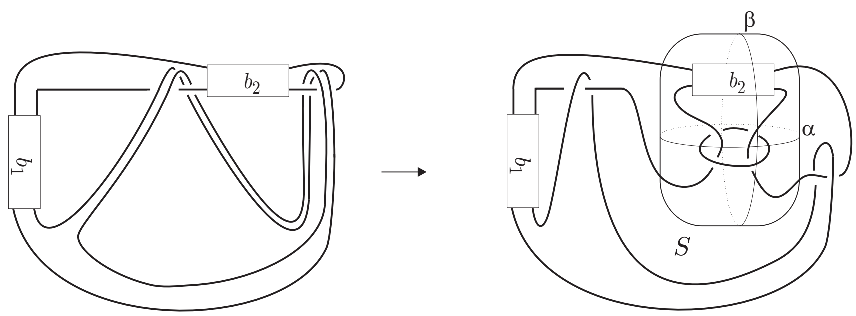

FIGURE 4. The link after 0-filling

two arcs and a single loop. Also, let M_i be the double branched cover of the 3-ball B_i branched over t_i.

Lemma 3.1.

(1) *M_1 is the exterior of the torus knot of type $(2, 2b_1 + 1)$. The loops α and β on S lift to a meridian μ and a regular fiber h of the exterior (with the unique Seifert fibration), respectively.*

(2) *M_2 is the union of the twisted I-bundle KI over the Klein bottle and the cable space C of type $(b_2, 1)$. The loop α lifts to a regular fiber of the cable space C with the unique Seifert fibration, and a regular fiber of KI with a Seifert fibration over the Möbius band.*

Proof. (1) By filling $\mathcal{T}_1$ with a rational tangle as in Figure 5, we obtain a trivial knot. Then the core γ of the filled rational tangle lifts to the torus knot of type $(2, 2b_1+1)$. This shows that M_1 is the exterior of the torus knot of type $(2, 2b_1+1)$, and α lifts to a meridian.

On the other hand, $\mathcal{T}_1$ is a Montesinos tangle whose double branched cover is a Seifert fibered manifold over the disk with two exceptional fibers. Moreover, β lifts to a regular fiber (see [5]).

(2) For $\mathcal{T}_2$, there is another essential Conway sphere P as illustrated in Figure 6. The inside of P is a Montesinos tangle, whose double branched cover is the twisted I-bundle KI over the Klein bottle. It is well known that KI admits two Seifert fibrations; one over the disk with two exceptional fibers, the other over the

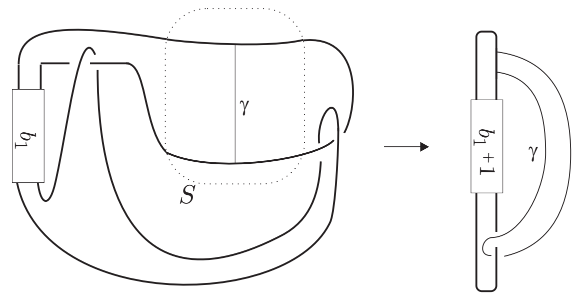

FIGURE 5. $\mathcal{T}_1$ filled with a rational tangle

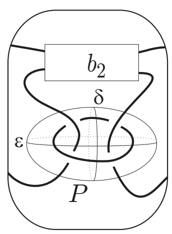

FIGURE 6. Conway sphere P in $\mathcal{T}_2$

Möbius band with no exceptional fiber. In fact, the loop δ (resp. ε) on P lifts to a regular fiber of the former (resp. latter) fibration.

The outside of P lifts to the cable space of type $(b_2, 1)$, where α lifts to a regular fiber with respect to its unique fibration. $\square$

Remark 3.2. In fact, M_2 admits a Seifert fibration over the Möbius band with one exceptional fiber of index $|b_2|$. Also, M_2 can be obtained by attaching a solid torus J to the twisted I-bundle KI over the Klein bottle along annuli on their boundaries so that a regular fiber on $\partial(KI)$, with a Seifert fibration over the Möbius band, runs $|b_2|$ times along a core of J.

Lemma 3.3. *For M_1, the fundamental group has a presentation $\pi_1(M_1) = \langle a, b : a^2 = b^{2b_1+1}\rangle$, with a meridian $\mu = b^{-b_1}a$ and a regular fiber $h = a^2 = b^{2b_1+1}$. Also, $\pi_1(M_2) = \langle x, y, z : x^{-1}yx = y^{-1}, y = z^{b_2}\rangle$.*

Proof. For M_1, it is a standard fact, see [5]. For M_2, we first have $\pi_1(KI) = \langle x, y : x^{-1}yx = y^{-1}\rangle$, where x^2 (resp. y) represents a regular fiber of KI with the Seifert fibration over the disk (resp. Möbius band). As in Remark 3.2, decompose M_2 into KI and a solid torus J along an annulus. Then $\pi_1(M_2) = \langle x, y, z : x^{-1}yx = y^{-1}, y = z^{b_2}\rangle$, where z represents a core of J (with a suitable orientation). $\square$

Proposition 3.4. *Let M be the resulting manifold by $4b_2$-surgery on K. Then the fundamental group $\pi_1(M)$ has a presentation*

$$\pi_1(M) = \langle x, y, z, a, b : x^{-1}yx = y^{-1}, y = z^{b_2}, a^2 = b^{2b_1+1}, \mu = y, h = zx^2\rangle,$$

where $\mu = b^{-b_1}a$ and $h = a^2 = b^{2b_1+1}$.

Proof. Let $\phi : \partial M_1 \to \partial M_2$ be the identification map. By Lemma 3.1, $\phi(\mu) = y$. Thus it is sufficient to verify that $\phi(h) = zx^2$.

Let D_0 be a disk with two holes, and let c_0 be the outer boundary component, and c_1, c_2 the inner boundary components. Then set $W = D_0 \times S^1$. We identify D_0 with $D_0 \times \{*\} \subset D_0 \times S^1$. See Figure 7, where W is obtained as the double branched cover of the left tangle.

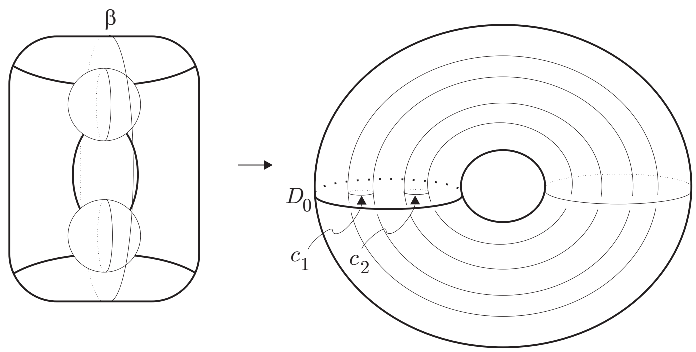

FIGURE 7. $W = D_0 \times S^1$

Let $T_i = c_i \times S^1$. Then M_2 is obtained from W by attaching a solid torus $S^1 \times D^2$ to T_1, KI to T_2. More precisely, c_2 is identified with a regular fiber of KI with the Seifert fibration over the disk. Similarly, c_1 is identified with $S^1 \times \{q\} \subset S^1 \times \partial D^2$.

Since c_0 is a lift of the loop β on S, $\phi(h) = c_0 = c_1 c_2$ with suitable orientations. As above, c_1 and c_2 correspond to z, x^2, respectively. □

4. LEFT-ORDERINGS

In this section, we prepare a few facts on left-orderings needed later.

Let G be a left-orderable non-trivial group. This means that G admits a strict total ordering $<$ such that $a < b$ implies $ga < gb$ for any $g \in G$. This is equivalent to the existence of a *positive cone* P ($\neq \varnothing$), which is a semigroup and gives a disjoint decomposition $P \sqcup \{1\} \sqcup P^{-1}$. For a given left-ordering $<$, the set $P = \{g \in G \mid g > 1\}$ gives a positive cone. Any element of P (resp. P^{-1}) is said to be *positive* (resp. *negative*). Conversely, given a positive cone P, declare $a < b$ if and only if $a^{-1}b \in P$. This defines a left-ordering.

We denote by $\mathrm{LO}(G)$ the set of all positive cones in G. This is regarded as the set of all left-orderings of G as mentioned above. For $g \in G$ and $P \in \mathrm{LO}(G)$, let $g(P) = gPg^{-1}$. This gives a G-action on $\mathrm{LO}(G)$. In other words, for a left-ordering $<$ of G, an element g sends $<$ to a new left-ordering $<^g$ defined as follows: $a <^g b$ if and only if $ag < bg$. We say that $<$ and $<^g$ are *conjugate* orderings. Also, a family $L \subset \mathrm{LO}(G)$ is said to be *normal* if it is G-invariant.

For $i = 1, 2$, let G_i be a left-orderable group and H_i a subgroup of G_i, and let $L_i \subset \mathrm{LO}(G_i)$ be a family of left-orderings. Let $\phi : H_1 \to H_2$ be an isomorphism. We call that ϕ is *compatible* for the pair (L_1, L_2) if for any $P_1 \in L_1$, there exists $P_2 \in L_2$ such that $h_1 \in P_1$ implies $\phi(h_1) \in P_2$ for any $h_1 \in H_1$.

Theorem 4.1 (Bludov-Glass [1]). *For $i = 1, 2$, let G_i be a left-orderable group and H_i a subgroup of G_i. Let $\phi : H_1 \to H_2$ be an isomorphism. Then the free product with amalgamation $G_1 * G_2$ ($H_1 \overset{\phi}{\cong} H_2$) is left-orderable if and only if there exist normal families $L_i \subset \mathrm{LO}(G_i)$ for $i = 1, 2$ such that ϕ is compatible for (L_1, L_2).*

The next is well known.

Lemma 4.2. *Consider a short exact sequence of groups*

$$1 \to K \to G \xrightarrow{\pi} H \to 1. \tag{4.1}$$

Suppose K and H are left-orderable, with left-orderings $<_H$ and $<_K$, respectively. For $g \in G$, declare that $1 < g$ if $\pi(g) \neq 1$ and $1 <_H \pi(g)$, or if $\pi(g) = 1$ and $1 <_K g$. Then this defines a left-ordering of G.

Suppose that we have a short exact sequence as (4.1), where H is torsion-free and abelian. Let A be a subgroup of G that is isomorphic to $\mathbb{Z}^2$. We assume that $A \cap K = \langle x \rangle$ is an infinite cyclic group. Since H is torsion-free, the element x is primitive in A, so we can choose another element y so that $\{x, y\}$ forms a basis of A.

Define two left-orderings $<_A$ and $<'_A$ of A as follows:

1. Given $x^r y^s \in A$, $1 <_A x^r y^s$ if $s > 0$, else $s = 0$ and $r > 0$.
2. Given $x^r y^s \in A$, $1 <'_A x^r y^s$ if $s > 0$, else $s = 0$ and $r < 0$.

Lemma 4.3. *With notation as above, there exists a normal family $L \subset \mathrm{LO}(G)$ of left-orderings such that every left-ordering of L restricts to either $<_A$ or $<'_A$ on the subgroup A.*

Proof. Choose a left-ordering $<_H$ of H such that $1 <_H \pi(y)$, and let $<_K$ be an arbitrary left-ordering of K. Construct a left-ordering $<$ of G as in Lemma 4.2, using $<_H$ and $<_K$. Then let $L \subset \mathrm{LO}(G)$ be the set of all conjugates of this ordering. By construction, L is normal.

Let $g \in G$ be an arbitrary element. For $x^r y^s \in A$ with $s \neq 0$, we have

$$1 <^g x^r y^s \iff 1 < g^{-1} x^r y^s g \iff 1 <_H \pi(g^{-1} x^r y^s g) = \pi(y)^s,$$

since H is abelian and $x \in K$. From the choice of $<_H$, this happens only when $s > 0$. Thus $<^g$ restricts to $<_A$ or $<'_A$ on A, according as $1 <_K g^{-1} x g$ or $g^{-1} x g <_K 1$. $\square$

Remark 4.4. The normal family L obtained in Lemma 4.3 contains both a left-ordering which restricts to $<_A$ on A and one which restricts to $<'_A$. For, if we have one, then the other is obtained by switching the positive cone and negative cone.

5. Proof of Theorem 1.1

Let $G_1 = \pi_1(M_1) = \langle a, b : a^2 = b^{2b_1+1} \rangle$, with a meridian $\mu = b^{-b_1} a$ and a regular fiber $h = a^2 = b^{2b_1+1}$. Then

$$\begin{aligned} G_1 &= \langle a, b, c : a^2 = b^{2b_1+1}, c = ba^{-1} \rangle \\ &= \langle b, c : b = cb^{2b_1} c \rangle. \end{aligned}$$

Thus this is Γ_{2b_1} in Navas's notation [13]. Hence, we can assign Navas's left-ordering to G_1.

In [16], we show that

Lemma 5.1. *Let $<^g$ be a conjugate ordering of Navas's left-ordering $<$ of G_1. Assume $1 <^g \mu^r h^s$. Then,*

(i) *$s > 0$; or*
(ii) *$s = 0$ and $r > 0$ (resp. $r < 0$) if $g^{-1} \mu g > 1$ (resp. $g^{-1} \mu g < 1$).*

Next, we will examine $G_2 = \pi_1(M_2) = \langle x, y, z : x^{-1}yx = y^{-1}, y = z^{b_2}\rangle$. Since G_2 is the fundamental group of an irreducible 3-manifold with toroidal boundary, it is left-orderable [3]. Let $\pi : G_2 \to \mathbb{Z}$ be a homomorphism defined by $\pi(x) = 1$, $\pi(y) = \pi(z) = 0$. Thus we have a short exact sequence

$$1 \to K \to G_2 \xrightarrow{\pi} \mathbb{Z} \to 1.$$

Let A be a rank two free abelian group generated by $\{y, zx^2\}$. In fact, $A = \pi_1(\partial M_2)$. Then $A \cap K = \langle y\rangle$. Hence by Lemma 4.3, we have a normal family $L \subset \mathrm{LO}(G_2)$ such that any left-ordering in L restricts to $<_A$ or $<' A$ on A, which are defined as follows:

(1) Given $y^r(zx^2)^s \in A$, $1 <_A y^r(zx^2)^s$ if $s > 0$, else $s = 0$ and $r > 0$.
(2) Given $y^r(zx^2)^s \in A$, $1 <'_A y^r(zx^2)^s$ if $s > 0$, else $s = 0$ and $r < 0$.

Proof of Theorem 1.1. Let $L_1 \subset \mathrm{LO}(G_1)$ be the set of all conjugate orderings of Navas's left-ordering of G_1. This is normal by definition. Let $L_2 \subset \mathrm{LO}(G_2)$ be the normal family given above.

Recall that the identification map $\phi : \partial M_1 \to \partial M_2$ is given by $\phi(\mu) = y$ and $\phi(h) = zx^2$. To show that $\pi_1(M)$ is left-orderable, it is sufficient to verify that ϕ is compatible for the pair (L_1, L_2) by Theorem 4.1.

For a left-ordering $<^g \in L_1$, suppose $1 <^g \mu^r h^s$. If $1 <^g \mu$, then $s > 0$, or $s = 0$ and $r > 0$ by Lemma 5.1. Since $\phi(\mu^r h^s) = y^r(zx^2)^s$, we choose a left-ordering in L_2, which restricts to $<_A$ on A. Similarly, if $\mu <^g 1$, then choose a left-ordering in L_2, which restricts to $<'_A$ on A. This shows that ϕ is compatible for (L_1, L_2). □

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